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1989 J. Phys. A: Math. Gen. 22 865

(http://iopscience.iop.org/0305-4470/22/7/017)

J. Phys. A: Math. Gen. 22 (1989) 865-878. Printed in the UK

Intermediate statistics of the quasi-energy spectrum and quantum localisation of classical chaos

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Received 29 April 1988

Abstract. The statistical properties of the quasi-energy spectrum in a simple quantum model are investigated for the case when the corresponding classical system is fully chaotic while quantum chaos is restricted by the localisation effects. It is shown that the level spacing distribution depends effectively on some parameter which is the ratio of the dimension of the eigenfunctions (mean localisation length) to the total number of the quasi-energy levels. Numerical data for a wide range of parameters of the system are given.

1. Introduction

The problem of properties of quantum systems whose classical counterparts reveal chaotic motion is still attractive for many scientists. One of the most important results in this field is the close relation between the spectral properties of quantum chaos and those of random matrices of certain symmetry [1-5]. This relation is far from being trivial if only for one reason: the quantum systems under consideration have no random parameters. Nevertheless, numerical experiments have shown that random matrix theory (RMT) can be successfully applied to describe statistical properties of the energy [4] (or quasi-energy [5]) spectrum as well as the chaotic structure of the eigenfunctions [6]. Specifically, the spacing distribution $P(s)$ of nearest-neighbour levels for such systems is described with high accuracy by a simple Wigner-Dyson supposition [7-9]:

$$P(s) = As^{\beta} \exp(-Bs^2) \quad (1)$$

where A and B are normalising constants, and β is a parameter depending on the symmetry of the system and characterising the repulsion between neighbour levels.

On the other hand, the so-called quantum localisation was discovered which can strongly supress chaos in a quantum system compared with a classical one [10, 11]. Such a localisation is analogous to the Anderson localisation in solid state physics but, in principle, is different because of the strongly deterministic nature of the system. As a result, it turns out that maximal quantum chaos appears under certain conditions when all eigenfunctions (EF) are random and fully extended (delocalised) in the restricted phase space of the system [5, 6]. It is clear that such a situation corresponds to the case of perturbation strong enough to cover all unperturbed states. Nevertheless, another case of so-called 'intermediate' quantum chaos is possible which is characterised by localised chaotic states of the system [12].

In this paper we study the spacing distribution $P(s)$ of nearest-neighbour levels taking into account the finite length of localisation of chaotic EF. Up until now much attention has been paid to the properties of quantum chaos. Nevertheless, correlation

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between the rate of quantum localisation and the statistical properties of the spectrum (see also [13, 14]) has not been investigated sufficiently.

It should be noted that the Berry-Robnik approach [15-17] to describe the spacing distribution $P(s)$ concerns a completely different situation for which the corresponding classical system is not fully chaotic and the deviation $P(s)$ from the Wigner-Dyson dependence (1) is caused by the existence of stable regions in phase space. It is known that in the other limiting case of completely integrable classical systems the level spacing distribution of quantum systems is very close to Poissonian $P(s) \sim \exp(-s)$ (for generic systems see [18-20]). For this reason the intermediate statistics in [15-17] is considered as the sum of two types of distribution (Poisson and Wigner-Dyson) depending on how the phase space of classical systems is divided into regions with stable and chaotic motion.

2. The kicked rotator model on the torus

Let us consider the well known kicked rotator (see e.g. [10, 11])

$$\hat{H} = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} + \varepsilon \cos\theta\, \delta_T(t) \qquad \delta_T(t) \equiv \sum_{m=-\infty}^{\infty} \delta(t - mT). \tag{2}$$

It is convenient to describe the motion of such a system by the mapping for the ψ function after one period T of perturbation

$$\psi(\theta, t+T) = \exp\left(\mathrm{i}\frac{T\hbar}{4I}\frac{\partial^2}{\partial\theta^2}\right) \exp\left(-\mathrm{i}\frac{\varepsilon}{\hbar}\cos\theta\right) \exp\left(\mathrm{i}\frac{T\hbar}{4I}\frac{\partial^2}{\partial\theta^2}\right)\psi(\theta, t). \tag{3}$$

It is written in a symmetric form where the ψ function is determined right in the middle of a free rotation, between two successive kicks. It is clear from (3) that the behaviour of the system depends only on two parameters: $\tau \equiv \hbar T/I$ and $k \equiv \varepsilon/\hbar$. It is known [21] that the corresponding classical system (the so-called 'standard mapping') has strongly chaotic motion under the condition $K \equiv k\tau \gg 1$.

According to numerical data (see e.g. [10, 11]) the quantum model (2), (3) imitates (under the additional condition $k \gg 1$ which means a large number of unperturbed levels covered by one kick) such a rough statistical property as the diffusion of energy in time and relaxation of the distribution function in momentum space. But it occurs only for some time $t \leqslant t^*$ after which the quantum interference effects start to influence more and more. As a result, for $t \gtrsim t^*$ classical diffusion is suppressed (and eventually stops (for generic irrational values $\tau/4\pi$)). It was established [11, 12] that this time t^*, corresponding to the classical diffusion ($E_{\mathrm{cl}} = Dt/2$), is determined by the rate of diffusion: $t^* \approx D \sim k^2$. The mechanism of this interesting effect is caused by the localisation of all eigenfunctions in unrestricted (infinite) momentum space of the system. The mean localisation length l_D of the EF, as was shown in [11, 12, 22], is related to the classical diffusion coefficient D by

$$l_D = \frac{D}{2} \approx \frac{k^2}{4}. \tag{4}$$

For model (2), (3) the level spacing distribution $P(s)$ must be Poissonian (see [5, 13]). This is related to the fact that the localisation length remains finite for any finite value of k and, therefore, the relative number of overlapped EF in infinite momentum space vanishes. Nevertheless, if we are interested in level statistics of those

EF which are overlapped, if only partly, then we can find some repulsion of nearest levels (see also [13]). It is natural to expect the rate of repulsion to be dependent on the rate of overlapping of the EF chosen from the total (infinite) number of states.

For the purpose of investigating the influence of localisation on statistical properties of quasi-energy spectra it is convenient to consider a model with a finite number N of levels:

$$\psi_n(t+T) = \sum_{m=1}^{N} U_{nm}(k, \tau)\psi_m(t) \qquad n, m = 1, 2, \ldots, N. \tag{5}$$

Here the finite unitary matrix U_{nm} determines evolution of any N-dimensional vector (Fourier transform of ψ function) of the system. It has the symmetric form

$$U_{nm} = G_{nn'}B_{n'm'}G_{m'm} \tag{6}$$

where the diagonal matrix $G_{ll'}$ corresponds to free rotation during a half period $T/2$:

$$G_{ll'} = \exp(\mathrm{i}(T/4)l^2)\delta_{ll'} \tag{7}$$

and matrix $B_{n'm'}$ describes the result of one kick:

$$B_{n'm'} = \frac{1}{2N+1}\sum_{l=1}^{2N+1}\left(\cos(n'-m')\frac{2\pi l}{2N+1} - \cos(n'+m')\frac{2\pi l}{2N+1}\right)\exp\left(-\mathrm{i}k\cos\frac{2\pi l}{2N+1}\right). \tag{8}$$

This model (5)-(8) with a finite number of states can be regarded as the quantum analogue of classical standard mapping on the torus with closed momentum p and phase θ. The difference of (5)-(8) from the model investigated in [5-6] is that the matrix U_{nm} describes only odd states of the system ($\psi(\theta) = -\psi(-\theta)$).

Such a model can be deduced from the model (2), (3) in the following way [5, 12]. Let us first consider (2), (3) for rational values of $\tau/4\pi = r/q$ (with r, q integers). It corresponds to the so-called quantum resonance [23, 24] for which all EF in a momentum representation are analogous to the Bloch states in a periodic crystal. Therefore, each EF is multiplied by phase factor $\exp(\mathrm{i}\theta_0)$ under the shift in period q. By selection of only periodic EF with $\theta_0 = 0$ in the model (2), (3) we can construct the finite matrix of size q which describes the evolution of periodic (in momentum space) states [5, 6]. The phase space of the corresponding classical model is closed in momentum p with the size $2\pi m_0$ where $m_0 = 2r$ comes from the periodicity in p. Then, selecting only odd states it is easy to pass to the matrix U_{nm} with the reduced size $N = (q-1)/2$ (here q is an odd number).

Our model (5), (6) can be, in principle, interpreted also as a model of some conservative system with a finite number of levels on the closed energy surface. Therefore, statistical properties of quantum chaos investigated here are typical also for autonomous systems with a chaotic counterpart in the classical limit. Similar models have also been considered in [14, 25].

3. Dimension of chaotic eigenstates: definition

Recently it was shown [5, 6] that under the conditions $K \gg 1$ (strong classical chaos) and $\Lambda \equiv l_D/N \gg 1$ (delocalisation of all EF of the system) the quantum chaos in model (5)-(8) is maximal. It means that statistical properties of the quasi-energy spectrum and chaotic structure of the EF are maximal. Specifically, the level spacing distribution

$P(s)$ for quasi-energies ω of U_{nm} is in excellent agreement with the dependence (1) for $\beta = 1$. Moreover, the distribution of the components of the EF in the unperturbed basis with a high accuracy corresponds to a microcanonical distribution of eigenvector components of finite random matrices [9]:

$$W_N(\varphi_n) = \frac{\Gamma(N/2)}{\sqrt{\pi}\,\Gamma((N-1)/2)}(1-\varphi_n^2)^{(N-3)/2} \qquad \sum_{n=1}^{N} \varphi_n^2 = 1. \tag{9}$$

As long as the matrix U_{nm} is unitary and symmetric the real and imaginary parts of its EF are equal to each other and equal to the EF of the real and imaginary part of U_{nm}. Therefore, the quantity φ_n in (9) is either the real or imaginary part of the EF of the matrix U_{nm}. Let us note that for $N \to \infty$, the microcanonical distribution (9) goes to a Gaussian one. It means that in the limit of large $N \gg 1$ all EF of the system with a maximal quantum chaos are Gaussian random functions.

In the opposite case, for $\Lambda < 1$, distribution $P(s)$ turns out to be intermediate between Wigner-Dyson (1) and Poissonian (see [5, 12]) and the eigenvectors of U_{nm} are random only on some localisation scale in momentum space.

In what follows we shall introduce a new definition of localisation length of the EF insofar as relation (4) has sense only for the model (2), (3) with infinite momentum space (or, similarly, for the model (5)-(8) with $l_D \ll N$). Unlike the traditional definition of localisation length as the inverse rate of decrease of amplitude of EF for $n \to \pm\infty$ (n is the number of unperturbed states) we determine l through the 'entropy' $\mathcal{H}$ of the EF (not to be confused with thermodynamic entropy):

$$\mathcal{H}_N^{(m)} = -\sum_{n=1}^{N} w_{nm} \log w_{nm} \qquad w_{nm} \equiv \varphi_{nm}^2. \tag{10}$$

Here m stands for the individual eigenvector of matrix U_{nm} ($m = 1, \ldots, N$).

In the limiting case of the microcanonical distribution of φ_{nm} the entropy $\mathcal{H}_N^{(m)}$ can be found from (10) using (9):

$$\mathcal{H}_N^{(m)} \approx \ln\left(\frac{N}{2}\alpha\right) + \frac{1}{N} \qquad N \gg 1 \tag{11}$$

where α is some constant

$$\alpha = \frac{4}{\exp(2-\gamma)} \approx 0.96 \tag{12}$$

with γ being the Euler constant ($\gamma \approx 0.577$). Now it can be seen that the quantity L_m

$$L_m \equiv \exp(\mathcal{H}_N^{(m)}) \tag{13}$$

means the effective number of components φ_{nm} with not too small values. As an example let us take the steady-state distribution $w_{nm} = 1/N$. Then the number L_m is equal to the maximal dimension of the EF ($L_m = N$). In comparison, for microcanonical distribution (9) we can get from (10), (11):

$$L_m \approx \alpha N/2. \tag{14}$$

This means that in spite of the ergodicity of the EF ($\langle\langle\varphi_{nm}^2\rangle\rangle = 1/N$) the fluctuations result in very small values $w_{nm} \approx 0$ for approximately half the components of φ_{nm}. This fact is related to the particular form of the distribution of w_{nm}, which is the χ^2 distribution with the divergence for $w_{nm} \to 0$. As a result, the probability density of EF turns out to be full of 'holes' both in momentum p and in 'coordinate' θ space.

Numerical data show that for $K = \text{constant} \gg 1$ the scale on which the EF can be considered random is less than the maximal dimension N and decreases with the quantum parameter k. Therefore, in accordance with (9)-(13) the mean localisation length can be associated with the average dimension d of the EF and determined by the 'entropy'

$$d \equiv \langle d_m \rangle = \frac{2}{\alpha} \langle L_m \rangle \approx 2\langle \exp(\mathscr{H}_N^{(m)}) \rangle \qquad d \gg 1 \tag{15}$$

where d is averaged over all eigenvectors of matrix U_{nm}.

In essence, relation (15) is a definition of both the mean localisation length and the dimension of chaotic EF. In the limit of maximal quantum chaos it gives $d = N$ but for $d \ll N$ numerical data show a good agreement with the usual definition of localisation length using the decay of EF on the 'tails' (see below).

It should be pointed out that for $d = N$ our matrix, in principle, is not a random one (it depends only on two dynamical parameters, K and k). Nevertheless, in this limiting case all statistical properties are very well described by random matrix theory (RMT). For $d < N$ the situation is much more difficult because RMT is not applicable. It seems that eigenvectors of U_{nm} with chaotic localised structure are isotropic only in some part of N-dimensional Hilbert space. It is interesting to see whether it is possible to develop a mathematical theory for such matrices.

4. The main properties of localised chaotic states: numerical data

We now investigate the dependence of dimension d on quantum parameter k in our model (5)-(8), when the classical parameter $K \approx 5$ is large enough to provide strong classical chaos [21]. All semiclassical conditions are supposed to be fulfilled: $N \gg 1$; $k \gg 1$; $\tau = 4\pi r/q \ll 1$; $(q = 2N+1)$. The result for $N = 398$ appears in figure 1 where

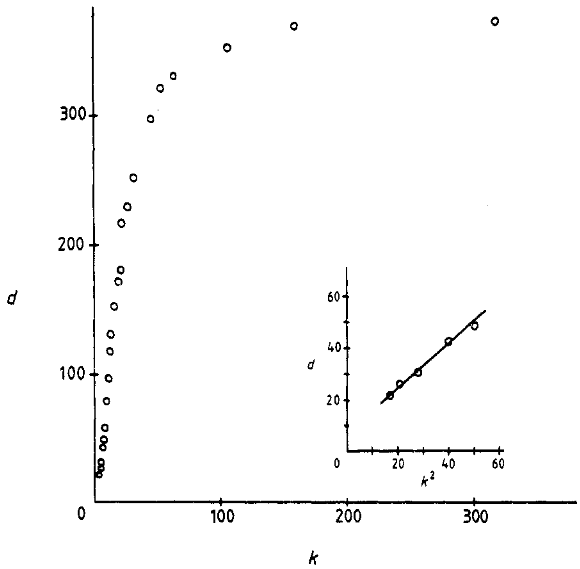


Figure 1. The mean localisation length (dimension d) against quantum parameter k for the fixed value of the classical parameter $K = 5$.

dimension d has been computed according to (10)-(15) with φ_n being the real part of all EF of matrix U_{nm}.

It is interesting to compare dimension $d(k)$ with localisation length l_D (see (4)) for small values of $d \ll N$. Numerical data for the free rotator model (2), (3) show [12, 22, 26] that localisation length measured by the decay rate of EF is equal to $l_D = k^2/4$ for $K=5$. As a rough estimate, let us suppose EF to be of the form $\varphi_n = l^{-1/2}\exp(-|n-n_0|/l)$ without taking into account fluctuations of its amplitude. Here n_0 is a centre of 'gravity' of the EF. Substituting this expression into (10), (15) we have, for $|n_0| \ll N$,

$$d \approx 2el \approx 5.4l \approx 1.25k^2 \tag{16}$$

while the fitted line in figure 1 (see insert) corresponds to $d \approx 0.87\,k^2$. This is quite a good correspondence of dimension $d \ll N$ to the common definition of localisation length. Nevertheless, further numerical experiments should be carried out not only in the region $k \gg 1$; $1 \ll d \ll N$ (see also [26]) but also for $k \gtrsim 1$ ($d \gtrsim 1$), which is slightly above the quantum stability border.

A remarkable property of localised chaotic states is large fluctuations of localisation length d_m of the individual EF. As an example, figure 2 represents localisation length distribution for three values of $k \approx 3.3$; 21.1; 317 (respectively, $r = 95$; 15; 1 for $\tau = 4\pi r/(2N+1)$), with the horizontal scale being the ratio of dimension d_m (localisation length $l_{\mathscr{H}}$) to the total number of levels N. It is seen that the largest fluctuations correspond to the value $d/N \approx 0.5$. In this case there are both strong localised states ($d_m \ll N$) and completely extended states ($d_m \approx N$). Nevertheless, in spite of these

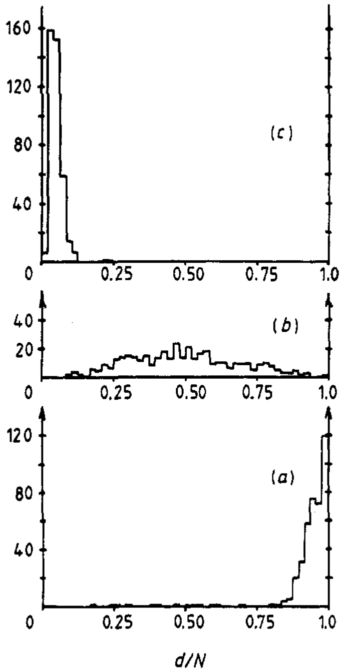


Figure 2. Three examples of the distribution of localisation length d_m for individual EF with different values of k and fixed $K=5$: (a) $r=1$, $k \approx 317$, $\beta \approx 0.95$; (b) $r=15$, $k \approx 21.1$, $\beta \approx 0.50$; (c) $r=95$, $k \approx 3.3$, $\beta \approx 0.05$.

fluctuations, the average dimension d can be described by a 'good' smooth dependence $d(k)$ (see figure 1).

Our approach to determine localisation length using 'entropy' of EF is closely associated with the simple idea of localisation length as an effective size on which the main probability of EF is concentrated. This is confirmed by the data in figure 3 where 'entropy' localisation length d is plotted against 'probability' localisation length l_w. The latter have been computed as a number of unperturbed states occupied by the 'main' part (95% probability) of the EF. There is a good correspondence between these two approaches in determining localisation length, especially when taking into account large fluctuations of individual EF.

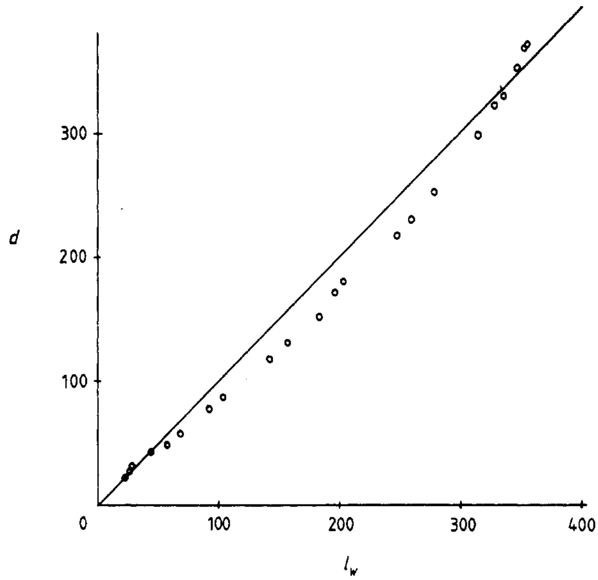


Figure 3. Relation between 'entropy' localisation length (dimension d) and 'probability' localisation length l_w.

5. Repulsion parameter β and analytical description of level spacing distribution

We can see that in the case of strong classical chaos the most essential parameter of quantum localisation is the average dimension of the EF. Therefore, it is natural to expect that statistical properties of the quasi-energy spectrum also effectively depend on the parameter $\beta = d/N$. In the limit $d \to N$ we have $\beta \to 1$ which corresponds to the Wigner-Dyson distribution for $P(s)$ with linear repulsion of neighbouring levels (confirmed numerically in [5]). In the other limiting case of integrable systems, for $\beta \to 0$, repulsion vanishes.

It is our conjecture that this parameter $\beta = d/N$ is a repulsion parameter even in the intermediate case of $0 < \beta < 1$. Then the problem of analytical description of the distribution $P(s)$ arises for the situation where all eigenstates are chaotic but not fully extended in the available phase space of the system. As far as we know, there is no good candidate for the analytical formula to describe this situation. For example, the above-mentioned Berry–Robnik dependence [15] has been derived for a completely different case of divided phase space of a classical system. The fitting parameter in this dependence has the meaning of the measure of chaotic regions compared with stable ones and cannot be used for our case. Another type of distribution (Brody distribution [27]) is also unsuitable because it has no physical support.

To obtain a proper analytical dependence of $P(s)$ we turn to RMT [7–9]. Our matrix U_{nm}, which has random properties in the limit case $l_D \gg N$ (see [5, 6]), is a unitary matrix. Therefore, it is natural to consider the variation of RMT for unitary random matrices, thoroughly developed by Dyson (see [7]). In his theory all statistical properties of spectra are determined by the joint distribution

$$Q(\omega_1, \ldots, \omega_N) = Q_0 \prod_{n \neq m} |e^{i\omega_n} - e^{i\omega_m}|^\beta \, d\omega_1, \ldots d\omega_N \tag{17}$$

of eigenangles ω_j which are related to eigenvalues $\lambda_j = \exp(i\omega_j)$ of a random unitary matrix of size $N \gg 1$. Here parameter β has meaning only for three cases: $\beta = 1$ stands for the ensemble of symmetric matrices, $\beta = 2$ for non-symmetric matrices and $\beta = 4$ for symplectic matrices.

Starting from (17), Dyson's approach gives the possibility to derive, in principle, the distribution for the spacing s between the neighbouring values ω_j located on the unit circle. This approach is based on the correspondence between the distribution of eigenangles ω_j of random unitary matrices and the steady-state distribution of two-dimensional Coulomb particles located on a ring (see [7]). In such a model β is an inverse temperature of Coulomb gas in the thermodynamic equilibrium. Therefore, in this physical analogy β varies from zero to infinity, but only for three values $\beta = 1; 2; 4$ is there a rigorous mathematical correspondence to random matrices. For other values of β, this correspondence fails and the question arises whether it is possible to find out the real physical situation where statistical properties of spectra are described by (17) with other (non-integer) values of β.

Our main conjecture is that the distribution (17) for non-integer β corresponds to quantum systems with a finite number of quasi-energy states under the condition that all eigenfunctions are chaotic and localised in the unperturbed basis. In our case we expect $0 \leqslant \beta \leqslant 1$ because both the system (2) and the model (5)–(8) are time-reversal invariant. Unfortunately, the question of deriving dependence $P(s)$ from (17) is far from trivial. It should be noted that even in the case of $\beta = 1; 2; 4$ there is no correct analytical formula for $P(s)$. It is known that the commonly used expression (1) is not related to RMT. Nevertheless in the main region $0 < s \leqslant 2$, this approximate dependence turns out to be very close to the exact one which stems from (17) (the latter have been obtained numerically with the use of Mehta's method [7, 8]). As long as the total number of levels does not, in practice, exceed several thousands, the Wigner–Dyson conjecture (1) is quite a good approximation (see [7]).

Here, as an approximate expression for $P(s)$ in the region $0 \leqslant \beta \leqslant 2$, the dependence

$$P(s) = As^\beta \exp\left[-\frac{\beta\pi^2}{16} s^2 - \left(C_0 - \frac{\beta}{2}\right)\frac{\pi}{2} s\right] \tag{18}$$

is suggested. Two normalised parameters A and C_0, in (18) are determined by the usual relations

$$\int_0^\infty P(s)\, ds = 1 \qquad \int_0^\infty sP(s)\, ds = 1 \tag{19}$$

where $s = 1$ is the mean distance between neighbouring levels. The dependence (18) written in the form which approximately takes into account the asymptotic expression of $P(s)$ for $s \to \infty$ has been obtained by Dyson [7]. On the other hand, it is quite close to (1) when $\beta = 1; 2$. In addition, for $\beta = 0$ the dependence (18) is Poissonian with the correct values of A and C_0. In figure 4 the expressions (18) and (1), together with the numerical data of RMT [7, 8], are shown. It is seen that the deviation of (18) does not exceed 5% for the most essential region $s \approx 1$–2 (from a practical point of view). It means that the dependence (18) can be regarded as a good approximation of (1) if the total number N of levels does not exceed $N \approx 10^4$. A much better correspondence occurs for $\beta = 2$ (see figure 5). Thus, our formula (18) is expected to be close to an exact (but unknown!) one, which stems from (17) with arbitrary values $0 \leqslant \beta \leqslant 2$.

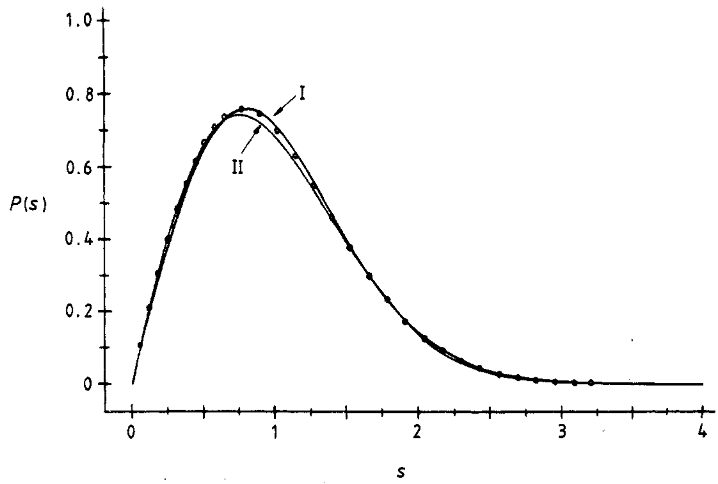


Figure 4. Distribution of the spacing between the neighbouring quasi-energy levels for $\beta = 1$; I, approximate Wigner-Dyson law (1); II, dependence (18); circles, numerical data for the 'true' dependence $P(s)$ in random matrix theory.

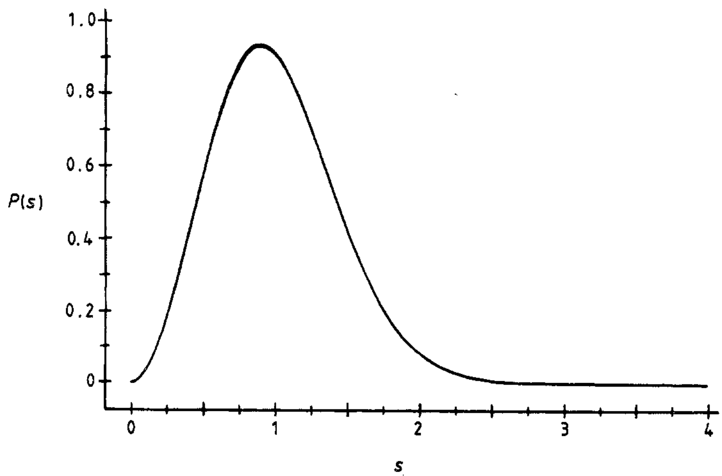
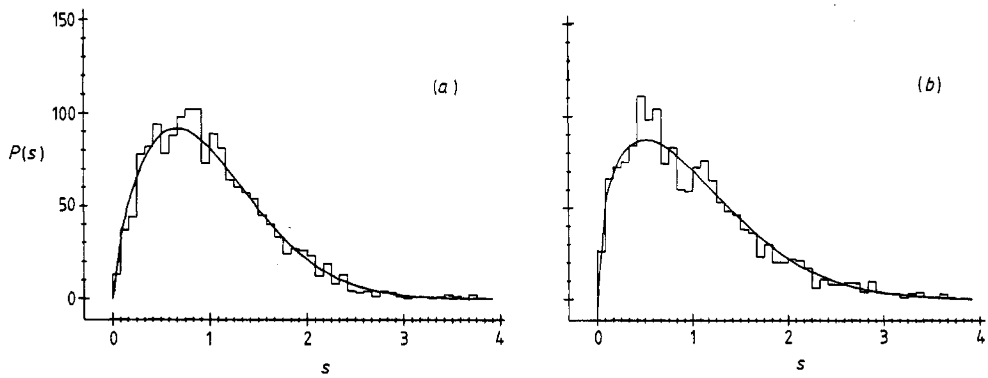


Figure 5. Analytical dependences $P(s)$ for $\beta = 2$. Both approximate curves (1) and (18) practically coincide (the discrepancy for $s \approx 1$ does not exceed 0.7%).

6. Numerical data for intermediate statistics ($0 < \beta < 1$)

We now come back to our question of intermediate statistics $P(s)$ for the model (5)–(8) dependent on the localisation of quantum chaos. Let us compare numerical data for $P(s)$ using the expression (18) where the parameter β is determined by the localisation length of chaotic localised EF through the expressions (10), (15). For this the dimension d of the EF of matrix U_{nm} and spacing distribution $P(s)$ for quasi-energies ω have been computed independently for a wide range of quantum parameter k. In all cases the classical parameter K was fixed ($K = 5$). To improve the statistics, the summing of $P(s)$ for four matrices U_{nm} of size $N = 398$ have been performed, with slightly different values of k ($\Delta k \ll k$). Quasi-energies ω_j have been found from the eigenvalues $\lambda_j = \exp(\mathrm{i}\omega_j)$ of matrix U_{nm}. To compute dimension d we use one of four matrices U_{nm}, averaging over all its EF.

Typical examples of $P(s)$ for three values $k \approx 39.8$; 21.1; 9.1 (respectively for $r = 8$; 15; 35) are given in figure 6. We can see good correspondence between numerical data and the dependence (18) with $\beta = d/N$. The χ^2_{23} value, for figure 6(a, b, c), is equal to 15.6; 27.2; 28.5; for 23 degrees of freedom with confidence levels 90%, 30% and 35%, respectively. More data are presented in figure 7 where it can be seen that the confidence level for all values of $\beta = d/N$ (circles) is not less than 5%. For the comparison, the χ^2 values are also given in figure 7 for two different relations between parameter β in (18) and parameter d/N. This was done as an additional control of our conjecture about linear dependence between the repulsion parameter β and the



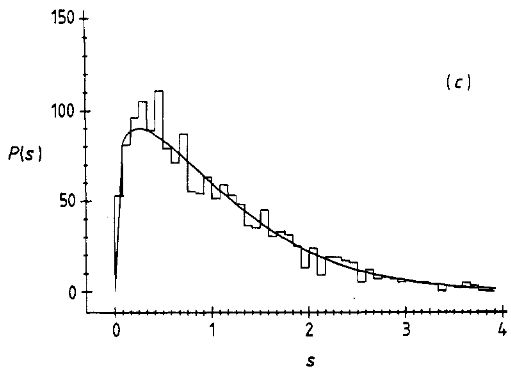


Figure 6. Three examples of intermediate statistics for $P(s)$ with the parameters $K = 5$, $N = 398$, and four matrices (6) with slightly different values of $k + \Delta k$ ($\Delta k \ll k$) are given. The total number of quasi-energy levels is equal to $M = 4N = 1592$. The staggered curve is numerical data, the smooth curve is the dependence (18) with $\beta = d/N$. (a) $k \approx 39.8$, $\beta \approx 0.76$, $\chi^2_{23} \approx 15.6$; (b) $k \approx 21.1$, $\beta \approx 0.48$, $\chi^2_{23} \approx 27.2$; (c) $k \approx 9.1$, $\beta \approx 0.22$, $\chi^2_{23} \approx 28.5$.

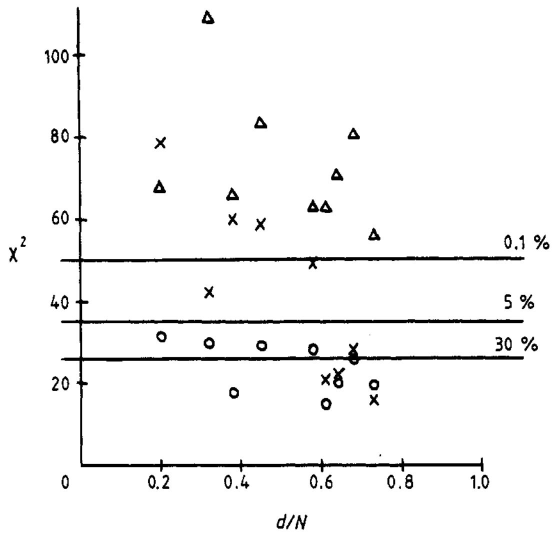


Figure 7. χ^2_{23} values for the comparison of numerical data with the dependence (18) are presented, with the total number of subintervals in s equal to 24. For β in (18) the circles are for $\beta = d/N$, triangles for $\beta = (d/N)^2$, crosses for $\beta = (d/N)^{1/2}$. χ^2_{23} values corresponding to 0.1%; 5%; 30% confidence levels are also given.

dimension d of chaotic EF. It is seen from figure 7 that linear dependence can be easily distinguished by the χ^2 approach.

A more accurate comparison of numerical data with the dependence (18) was now carried out with β as a fitting parameter. For this, the most suitable values of β (circles in figure 8) have been computed, which correspond to the minimum χ^2 value, together with the deviations in β corresponding to the 5% confidence level. As a result, we can see that all the data are well described by (18) with linear dependence between β

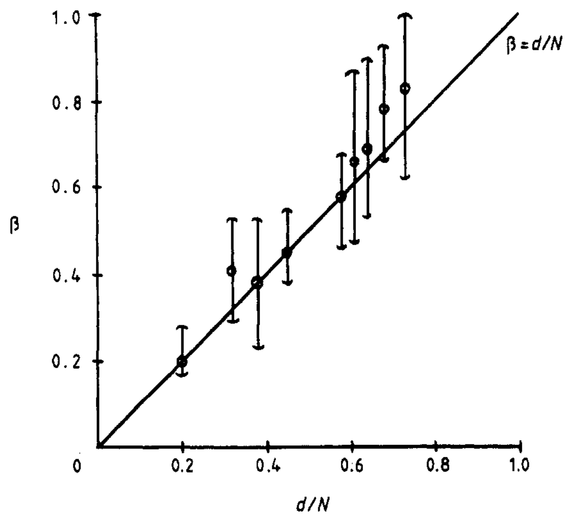


Figure 8. The fitting parameter β in (18) as a function of relative dimension d/N of the EF. Circles are values of β corresponding to the minimum value of χ^2_{23}; the bars indicate the 5% confidence level.

and d/N. It is clear from figure 8 that the spread in β is decreasing with β. This means that when the distribution $P(s)$ is approaching a Poissonian form it is becoming more sensitive to the analytical form of $P(s)$.

7. Level spacing distribution without time-reversal symmetry: discussion

It was shown above that intermediate statistics $P(s)$ for classically chaotic systems with quantum localisation of EF can be well approximated by the distribution (18) where β is the ratio of EF dimension to the total number N of states. Our numerical data are given for not too small values of $\beta \gtrsim 0.2$. As for $\beta \ll 1$, computer experiments are very difficult because in this case it is necessary to increase the size of matrix U_{nm}. This relates not only to the fact that the quantum parameter k must exceed the Shuryak border [28] ($k \gg k_{cr} \approx 1$; see [10-12]) but also to the condition $d \sim k^2 \gg 1$. The latter means that the dimension of the EF should be large enough for the EF to be regarded as chaotic on the localisation scale.

It is important to note that the meaning of the parameter β can be generalised to the values $\beta > 1$. Indeed, for the unitary ensemble of random matrices ($\beta = 2$ in (1)) the maximal dimension of chaotic states is equal to $2N$. This relates to the fact that each EF now has not N but $2N$ independent components (real and imaginary parts) in this case. Analogously, we have $\beta = d/N = 4$ for the symplectic (see [7-9]) ensemble as long as each eigenvector is determined by $4N$ independent random components.

In particular, it can be concluded from the above that for systems which are not time-reversal invariant, the value of β is not restricted to $\beta = 1$. For such systems the limiting quantum chaos corresponds to $\beta = 2$ and statistical properties of spectra are described by RMT for the unitary ensemble (see examples in [5, 25]). Then according to our approach, the spacing distribution $P(s)$ in the case of intermediate statistics will be described by the same dependence (18) with $\beta = d/N$.

For the preliminary test of this statement our model (5)-(8) was modified in such a way that the time-reversal invariance was broken (see [5] for details). As a result, the new matrix $\tilde{U}_{nm}$ turned out to be non-symmetrical. Therefore, real and imaginary parts of the EF in the unperturbed ($k = 0$) basis must be independent. It was shown in [5] that this can be done by choosing (instead of $\cos\theta$) the perturbation with broken symmetry (under transformation $\theta \to -\theta$) and by adding to the unperturbed spectrum the linear dependence in momentum n. The latter modification is analogous, in essence, to switching on the magnetic field.

As a result, the new matrix $\tilde{U}_{nm}$ takes the form

$$\tilde{U}_{nm} = \frac{1}{N} \exp[\tfrac{1}{4}\mathrm{i}\tau(n^2 + \xi n)] \sum_{p=-N_1}^{N_1} [\mathrm{i}k\cos(2\pi p/N + \eta)] \times \exp\left(\mathrm{i}\frac{2\pi}{N} p(n-m)\right) \exp[\tfrac{1}{4}\mathrm{i}\tau(m^2 + \xi m)] \tag{20}$$

where $N = 2N_1 + 1$; $n, m = -N_1, \ldots, N_1$. In the numerical simulation the values of parameters are equal to $\tau = 4\pi \times 16/N$, $N = 199$, $\xi \approx 1.88$; $\eta \approx 0.81$. The strength of perturbation k was chosen in such a way that dimension d is to be equal to $d \approx N$. The quantity d was numerically found according to the formulae (10), (15) with the only exception that the sum (10) runs over both real parts and imaginary parts of the EF. Therefore, the total number of components in the sum (10) is equal to $2N$ with the usual normalisation

$$\sum_{n=1}^{2N} w_n = 1 \qquad w_n = \begin{cases} (\mathrm{Re}\ \phi_n)^2 & n=1,\ldots,N \\ (\mathrm{Im}\ \phi_n)^2 & n=N+1,\ldots,2N. \end{cases} \tag{21}$$

It is clear that our new matrix $\tilde{U}_{nm}$ describes the evolution of any state of the system, unlike the matrix U_{nm} which was obtained for odd states ($\psi(\theta)=-\psi(-\theta)$) of the model (5)–(8).

The result of this simulation is presented in figure 9. Here the numerical data are the sum over the distributions $P(s)$ for eight matrices $\tilde{U}_{nm}$ with slightly different values of k in the interval $13.0 \leqslant k \leqslant 13.9$. The matrix size is equal to $N=199$; therefore, the total number of quasi-energy levels is equal to $M = 8 \times N = 1592$. It is seen from figure 9 that the correspondence between the numerical data and the dependence (18) is good. It should be pointed out that for the model (20), the distribution $P(s)$ in figure 9 is intermediate between Poissonian and Wigner–Dyson (1) with $\beta=2$. In the limiting case of large $l_D \gg N$ the distribution $P(s)$ was shown numerically in [5] to be in very good correspondence with the prediction of RMT for a Gaussian unitary ensemble ($\beta = 2$ in (1)).

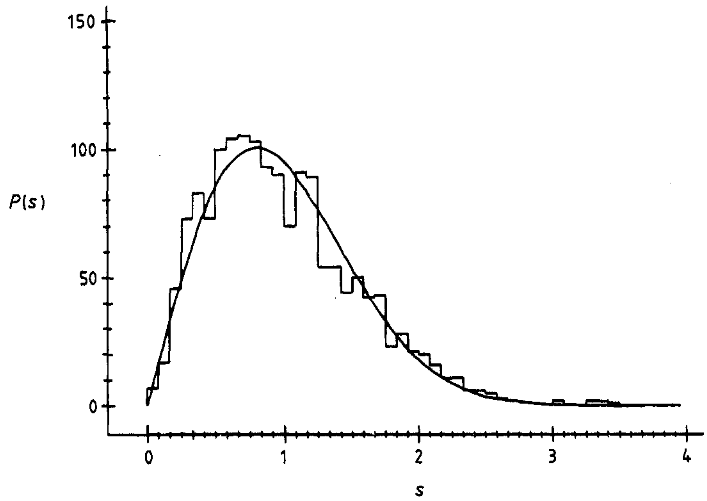


Figure 9. The intermediate statistics of $P(s)$ for the model (20) which is not time-reversal invariant. The size of the matrix $\tilde{U}_{nm}$ is equal to $N=199$; the quantum parameter k for eight matrices $\tilde{U}_{nm}$ changes in the interval $k = 13.0$–13.9 which corresponds, approximately, to $d \approx N$ for the average dimension of the EF. The smooth curve is the analytical dependence (18) with $\beta = 1$. The χ^2 approach gives $\chi^2_{23} \approx 31$ with $\approx 10\%$ confidence level.

Acknowledgments

The author is greatly indebted to B V Chirikov for suggestions and encouragement at all stages of this work and to V G Zelevinsky, V V Sokolov and D D Shepelyansky for fruitful discussions.

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